Homogenization Results for a Deterministic Multi-domains Periodic Control Problem

Nicoletta Tchou, Guy Barles, Ariela Briani, Emmanuel Chasseigne

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TOURS-HJnet-Netco

Nicoletta Tchou
with G. Barles, A. Briani and E. Chasseigne

Université de Rennes 1

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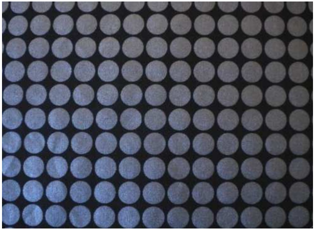
Plan

Geometry

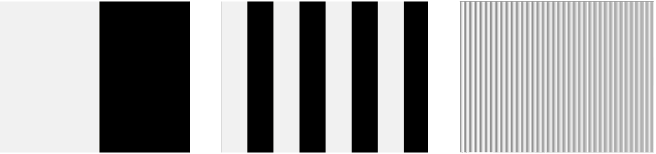
- $\mathbb{R}^N = \Omega_1 \cup \Omega_2 \cup \mathcal{H}$
- Ω_1, Ω_2 are open subsets of $\mathbb{R}^N$
- $\Omega_1 \cap \Omega_2 = \emptyset$
- $\mathcal{H} = \partial\Omega_1 = \partial\Omega_2$ is regular
- Ω_i are $\mathbb{Z}^N$-periodic, i.e. $x \in \Omega_i \Leftrightarrow x + z \in \Omega_i$, $\forall z \in \mathbb{Z}^N$.



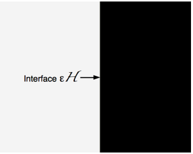
Hamilton-Jacobi equations in $\varepsilon\Omega_i$

$$\begin{aligned} \lambda u_\varepsilon(x) + H_1(x, \frac{x}{\varepsilon}, Du_\varepsilon(x)) = 0 \quad & \text{in } \varepsilon\Omega_1 , \\ \lambda u_\varepsilon(x) + H_2(x, \frac{x}{\varepsilon}, Du_\varepsilon(x)) = 0 \quad & \text{in } \varepsilon\Omega_2 , \end{aligned} \tag{1.1}$$

- $\varepsilon \searrow 0$
- $\lambda > 0$
- $H_i(x, y, p) := \sup_{\alpha_i \in A_i} \{-b_i(x, y, \alpha_i) \cdot p - l_i(x, y, \alpha_i)\}$
  $x \in \mathbb{R}^N,\ y \in \overline{\Omega}_i,\ p \in \mathbb{R}^N.$



Hamilton-Jacobi equations on $\varepsilon\mathcal{H}$: Ishii conditions



$$\min\{\lambda u_\varepsilon(x) + H_1(x, \frac{x}{\varepsilon}, Du_\varepsilon(x)), \lambda u_\varepsilon(x) + H_2(x, \frac{x}{\varepsilon}, Du_\varepsilon(x))\} \leq 0 \quad \text{on } \varepsilon\mathcal{H} \,,$$

$$\max\{\lambda u_\varepsilon(x) + H_1(x, \frac{x}{\varepsilon}, Du_\varepsilon(x)), \lambda u_\varepsilon(x) + H_2(x, \frac{x}{\varepsilon}, Du_\varepsilon(x))\} \geq 0 \quad \text{on } \varepsilon\mathcal{H} \,, \tag{1.2}$$

- (1.1)-(1.2) is NOT a well-posed problem
- U_ε^+:=maximal sub-solution
- U_ε^-:=minimal super-solution

Assumptions

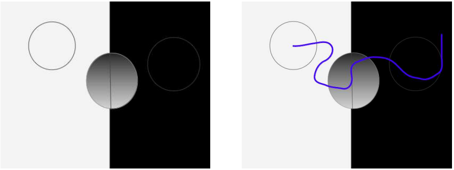
- Standard assumptions on costs and dynamics in Ω_i
- Convexity assumption:

$$\cup_{\alpha_i \in A_i} (b_i(x, y, \alpha_i), l_i(x, y, \alpha_i)) \text{ is convex}$$

- Strong controllability assumption:

$$\mathbf{B}_i(x, y) := \left\{ b_i(x, y, \alpha_i) : \alpha_i \in A_i \right\} \supset \{|z| \leq \delta\}, \quad \delta > 0$$

Optimal control characterization of extremal Ishii-solutions : Trajectories



$$\dot{X}^{\varepsilon}_{x_0}(t) \in \mathcal{B}\left(X^{\varepsilon}_{x_0}(t), \frac{X^{\varepsilon}_{x_0}(t)}{\varepsilon}\right) \quad \text{a.e. } t \in (0, +\infty)\,; \quad X^{\varepsilon}_{x_0}(0) = x_0$$

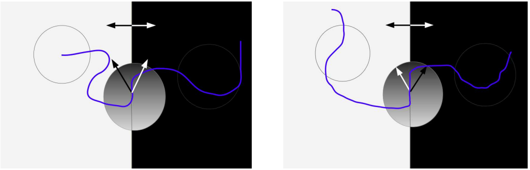
$$\mathcal{B}(x,y) := \begin{cases} \mathbf{B}_1(x,y) & \text{if } y \in \Omega_1 \\ \mathbf{B}_2(x,y) & \text{if } y \in \Omega_2 \\ \overline{\text{co}}\big(\mathbf{B}_1(x,y) \cup \mathbf{B}_2(x,y)\big) & \text{if } y \in \mathcal{H} \end{cases}$$

Dynamics on the interface: $y \in \mathcal{H}$, $a = (\alpha_1, \alpha_2, \mu)$ $\alpha_i \in A_i$, $\mu \in [0,1]$

$$b_{\mathcal{H}}(x,y,a) := \mu b_1(x,y,\alpha_1) + (1-\mu) b_2(x,y,\alpha_2)$$

Optimal control characterization of extremal Ishii-solutions : regular and singular trajectories

$$b_{\mathcal{H}}(x,y,a) := \mu b_1(x,y,\alpha_1) + (1-\mu) b_2(x,y,\alpha_2)$$



REGULAR SINGULAR

- tangential : $b_{\mathcal{H}}(x,y,a)\cdot \mathbf{n}_i(y) = 0$
- REGULAR: tangential and $b_i(x,y,\alpha_i)\cdot \mathbf{n}_i(y) \geq 0 \ \forall i = 1,2$
- SINGULAR: tangential and $b_i(x,y,\alpha_i)\cdot \mathbf{n}_i(y) \leq 0 \ \forall i = 1,2$

$\mathbf{n}_i(y)$ unitary normal exterior vector to Ω_i in y

Control characterization of extremal Ishii solutions: tangential Hamiltonians

Dynamics and cost at the interface $y \in \mathcal{H}$:

- $b_{\mathcal{H}}(x,y,a) := \mu b_1(x,y,\alpha_1) + (1-\mu) b_2(x,y,\alpha_2)$
- $l_{\mathcal{H}}(x,y,a) := \mu l_1(x,y,\alpha_1) + (1-\mu) l_2(x,y,\alpha_2)$

Tangential Hamiltonians:

- $H_T(x,y,p) := \sup_{a \in A_0(x,y)} \{ -< b_{\mathcal{H}}(x,y,a), p > - l_{\mathcal{H}}(x,y,a) \}$
- $H_T^{\mathrm{reg}}(x,y,p) := \sup_{a \in A_0^{\mathbf{reg}}(x,y)} \{ -< b_{\mathcal{H}}(x,y,a), p > - l_{\mathcal{H}}(x,y,a) \}$

$$H_T(x,y,p) \geq H_T^{\mathrm{reg}}(x,y,p)$$

Proposition

The tangential Hamiltonians are as regular as the data

Infinite horizon optimal control problems

$$\text{Cost} = J(x_0; (X^\varepsilon_{x_0}, a)) := \int_0^{+\infty} l\Big(X^\varepsilon_{x_0}(t), \frac{X^\varepsilon_{x_0}}{\varepsilon}(t), a(t)\Big) e^{-\lambda t} dt$$

$$\begin{aligned} l\Big(X^\varepsilon_{x_0}(t), \frac{X^\varepsilon_{x_0}}{\varepsilon}(t), a(t)\Big) := {} & l_1\Big(X^\varepsilon_{x_0}(t), \frac{X^\varepsilon_{x_0}}{\varepsilon}(t), \alpha_1(t)\Big)\mathbf{1}_{\mathcal{E}_1}(t) \\ & + l_2\Big(X^\varepsilon_{x_0}(t), \frac{X^\varepsilon_{x_0}}{\varepsilon}(t), \alpha_2(t)\Big)\mathbf{1}_{\mathcal{E}_2}(t) \\ & + l_{\mathcal{H}}\Big(X^\varepsilon_{x_0}(t), \frac{X^\varepsilon_{x_0}}{\varepsilon}(t), a(t)\Big)\mathbf{1}_{\mathcal{E}_{\mathcal{H}}}(t) \end{aligned}$$

Value functions:

$$V^-_\varepsilon(x_0) := \inf_{(X^\varepsilon_{x_0}, a) \in \mathcal{T}^\varepsilon_{x_0}} J(x_0; (X^\varepsilon_{x_0}, a))$$

$$V^+_\varepsilon(x_0) := \inf_{(X^\varepsilon_{x_0}, a) \in \mathcal{T}^{\mathrm{reg},\varepsilon}_{x_0}} J(x_0; (X^\varepsilon_{x_0}, a))$$

$$V^-_\varepsilon(x_0) \le V^+_\varepsilon(x_0)$$

Known facts: BBC 2013-2014

Theorem

V_ε^- and V_ε^+ are viscosity solutions of the Hamilton-Jacobi-Bellman equations (1.1)-(1.2).
V_ε^- is a subsolution of $\lambda v(x) + H_T(x, \frac{x}{\varepsilon}, D_{\mathcal{H}} v(x)) \leq 0$ for $x \in \varepsilon\mathcal{H}$.

Theorem

- $V_\varepsilon^- = U_\varepsilon^-$
- $V_\varepsilon^+ = U_\varepsilon^+$
- global (on $\mathbb{R}^N$) (and local) strong comparison principle for
  - Lipschitz continuous subsolutions satisfying $\lambda v(x) + H_T(x, \frac{x}{\varepsilon}, D_{\mathcal{H}} v(x)) \leq 0$
  - lsc supersolution

Notation

$\lambda v(\cdot) + \mathbf{H}^-(\cdot, \cdot, Dv(\cdot)) = 0 \Leftrightarrow$ (1.1)-(1.2) *and* $\lambda v(\cdot) + H_T(\cdot, \cdot, D_{\mathcal{H}} v(\cdot)) \leq 0$

Plan

Theorem (The cell problem for U_ε^-)

$\forall x \in \mathbb{R}^N$, $\forall p \in \mathbb{R}^N$ (frozen) $\exists!$ $C := \bar{H}^-(x,p)$ such that

$$\mathbf{H}^-(x, y, Dv(y) + p) = C$$

has a Lipschitz continuous, $\mathbb{Z}^N$-periodic viscosity solution V^-.

Proof.

Classical, uses a stability result for the existence and the comparison result for $\mathbf{H}^-$ for the uniqueness (ρ-**problem**)... □

Proposition

$\bar{H}^-(x,p)$ is regular (Lipschitz and coercive) and a comparison principle for the effective problem holds true.

Theorem (Main theorem for U_ε^-)

$(U_\varepsilon^-)_{\varepsilon>0}$ converges locally uniformly to U^- the unique solution of

$$\lambda U^-(x) + \bar{H}^-(x, DU^-) = 0$$

Proof The classical methods of Lions-Papanicolaou-Varadhan and Evans-perturbed test function apply (the local comparison principle for $\mathbf{H}^-$ is an essential tool). $\square$

Plan

Theorem (The cell problem for U_ε^+)

$\forall x, p \in \mathbb{R}^N$ (*frozen*) $\exists!$ $\bar{H}^+(x,p) \in \mathbb{R}$ such that $\exists V^+$ Lipschitz continuous, periodic function $\forall \tau \geq 0$, $y_0 \in \mathbb{R}^N$

$$(*) V^+(y_0) = \inf_{(Y_{y_0},a)\in\mathcal{T}^{\mathrm{reg}}_{y_0}} \{ \int_0^\tau (l(x, Y_{y_0}(t), a(t)) + b(x, Y_{y_0}(t), a(t)) \cdot p + \bar{H}^+(x,p))dt + V^+(Y_{y_0}(\tau)) \}$$

Proof.

Dynamic Programming Principle for the existence and uniqueness follows from the definition $(*)$ and regularity of V^+ □

Proposition

$\bar{H}^+(x,p)$ is regular (Lipschitz and coercive) and a comparison principle for the effective problem holds true.

Proof. Characterisation of the solution of the approximate cell problem as maximal subsolution.□

Theorem (Main theorem for U_ε^+)

$(U_\varepsilon^+)_{\varepsilon>0} \to U^+$ unique viscosity solution of

$$\lambda u(x) + \bar{H}^+(x, Du(x)) = 0 \text{ in } \mathbb{R}^N.$$

Proof.

U^+ is a supersolution:
EDP proof as in the U^- case thanks to a local property of maximality.
U^+ is a subsolution:
a comparison principle concerning supersolutions and U_ε^+ does not hold

- First case: $b_i(x, y, a) = b_i(y, a)$, $l_i(x, y, a) = l_i(y, a)$.
  Main tool: the perturbed test function *is almost super-optimal*
- The general case:
  - Trajectories in the definition of correctors: $\dot{Z}_\varepsilon(t) \in \mathcal{B}\left(\bar{x}, \frac{Z_\varepsilon(t)}{\varepsilon}\right)$
  - Trajectories in the homogenization problem: $\dot{X}_\varepsilon(t) \in \mathcal{B}\left(X_\varepsilon(t), \frac{X_\varepsilon(t)}{\varepsilon}\right)$
- Approximating problems with dynamics constant in the slow variable

□

Plan

The 1-D case: an example when $\bar{H}^+(x,0) \neq \bar{H}^-(x,0)$

$$\Omega_1 = \bigcup_{k\in\mathbb{Z}}]2k, 2k+1[\qquad \Omega_2 = \bigcup_{k\in\mathbb{Z}}]2k+1, 2k+2[$$

$$b_1(x,y,\alpha_1) = \alpha_1,\ b_2(x,y,\alpha_2) = \alpha_2,\ A_1 = A_2 = [-1,+1]$$

$$l_1(x,y,\alpha_1) = |\alpha_1 - \cos(\pi y)| + 1 - |\cos(\pi y)|$$

$$l_2(x,y,\alpha_2) = |\alpha_2 + \cos(\pi y)| + 1 - |\cos(\pi y)|.$$

The data are independent of the x variable, then U^+ and U^- are constants and only $p = 0$ is relevant in $\bar{H}^{\pm}$.
Ergodic charactecterisation implies:

$$\bar{H}^{\pm} = \lim_{t\to+\infty}\left(- \inf_{(Y_0,a)\in\mathcal{T}_0^{\bigcirc}} \left\{ \frac{1}{t}\int_0^t l\big(Y_0(s), a(s)\big)\,ds \right\} \right).$$

The best strategy is "singular" $\alpha_1 = 1$ $\alpha_2 = -1$ and cannot be used in the $\bar{H}^+$-case. Finally $\bar{H}^- = 0$, $\quad \bar{H}^+ < 0$

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